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Classification of Some Metric Automorphisms Defined by Standish

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Let F be a finite set with a probability distribution $\{P_i : i \in F\}$ and $(\Omega, \mathscr{F}, P)$ denote the product space of countably many copies of (F, P). A permutation (bijection) ϕ of the integers induces an invertible measure preserving transformation T_ϕ on $(\Omega, \mathscr{F}, P)$ given by the equation $(T_\phi w)_j = w_{\phi(j)}$. Such metric automorphisms we call S-automorphisms.

We show in this paper that S-automorphisms are ergodic if and only they are Bernoulli shifts and two ergodic S-automorphisms are isomorphic if and only if their associated permutations are conjugate.

We also show that S-automorphisms have discrete spectrum if and only if they have zero entropy and every S-automorphism is either a Bernoulli shift, has discrete spectrum, or is a product of a Bernoulli shift and an automorphism with discrete spectrum.

S-automorphism with discrete spectrum are those whose associated permutations contain only cycles of finite length. These automorphisms are studied according to the number of such finite cycles. Those whose permutations have infinitely many finite cycles with unbounded lengths are shown to be antiperiodic and those whose permutations have infinitely many finite cycles of bounded length are periodic with almost no fixed points. An example is given of two automorphisms of this latter type which are isomorphic but whose permutations are not conjugate.

A complete isomorphism invariant is given for S-automorphisms whose associated permutations consist of finitely many finite cycles. Using this invariant we show that if ϕ is either a product of k disjoint cycles of prime power p^α, or a single cycle of length pq where p and q are primes, or a product of k disjoint cycles of prime lengths $p_1 < p_2 < \cdots < p_k$ and if ψ is a permutation such that T_ψ and T_ϕ are isomorphic then ψ is conjugate to ϕ.

In this paper we investigate a class of measure preserving transformations defined by Standish [8] and independently by Steinhaus [9]. The transformations are obtained by shuffling the components of points (sequences) from the product of a countable infinitude of copies of a finite set with a given probability distribution. These automorphisms we call S-automorphisms. The shuffling is determined by a permutation of the integers and in certain cases it is shown that two such automorphisms are isomorphic as metric automorphisms if and only if the underlying permutations are conjugate as permutations. This is the case

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if we restrict our consideration to ergodic automorphisms. Thus ergodic S-automorphisms are metrically isomorphic if and only if they are topologically isomorphic. We conjecture that the same is true for S-automorphisms that are associated with permutations which consist of finitely many disjoint cycles of finite length. It is shown that for certain types of nonergodic automorphisms T_ϕ if T_ψ is isomorphic to T_ϕ then ψ must be conjugate to ϕ. An example is given of two nonconjugate permutations ϕ and ψ for which T_ϕ and T_ψ are isomorphic.

1. Ergodic S-Automorphisms and Entropy

Let F be a finite set and $\{p_i: i \in F\}$ be a probability distribution on F. Denote the product space $\bigtimes_{j=-\infty}^{+\infty} F_j$, where $F_j = F$, by Ω and let μ be the product measure obtained from the distribution $\{p_i\}$.

1.1 Definition. For each permutation ϕ of the integers (i.e., 1 to 1 map of the integers onto themselves) define T_ϕ to be the measure preserving transformation on (Ω, μ) defined by $(T_\phi w)_n = w_{\phi(n)}$. The class of all such metric automorphisms will be denoted by $\mathscr{S}$ and elements of $\mathscr{S}$ will be called S-automorphisms.

Standish [8] has shown that an S-automorphism T_ϕ is ergodic if and only if the underlying permutation ϕ has only infinite cycles. It is easy to see that if ϕ has only a finite number, say k, of infinite cycles then $T = T_1{}^k$ where T_1 is a Bernoulli shift. Thus T is a Bernoulli shift and $h(T) = kh(T_1)$ where h denotes the entropy. Also if ϕ has infinitel many infinite cycles then T_ϕ is the increasing limit of a sequence of Bernoulli shifts as can be seen as follows.

Let the disjoint cycles of ϕ_ϕ be $\{\phi_j: j = 1, 2, \ldots\}$. Define the partition $P^{(k)}$ by

$$P^{(k)} = \{p^{(k)}(i_1, \ldots, i_k): i_j \in F\},$$

where $p^{(k)}(i_1, \ldots, i_k) = \{w \in \Omega: w_{n_1} = i_1, \ldots, w_{n_k} = i_k\}$ and $n_1, n_2, \ldots, n_k$ are selected from disjoint cycles $\phi_1, \ldots, \phi_k$. Then $\{T_\phi{}^j P^{(k)}\}_{j=-\infty}^{+\infty}$ is an independent family of finite partitions for each k and $(T_\phi, V_{j=-\infty}^{+\infty} T_\phi{}^j P^{(k)})$ is a Bernoulli shift. Since $P^{(k+1)}$ refines $P^{(k)}$, and ϕ has no fixed points, $V_{j=-\infty}^{\infty} T_\phi{}^j P(k)$ increases in k to the total σ-algebra and T_ϕ is a Bernoulli shift by Ornstein's results [6]. Also $h(T_\phi, V_{j=-\infty}^{\infty} T_\phi{}^j P^{(k)}) = -k \sum p_i \log p_i$ so that $h(T_\phi) = \lim_{k\to\infty} h(T_\phi, V_{-\infty}^{\infty} T_\phi{}^j p^{(k)}) = \infty$.

1.2 Theorem. *If ϕ consists entirely of infinite cycles then T_ϕ is a Bernoulli shift and $h(T_\phi) = -k \sum p_i \log p_i$ where k is the number of cycles.*

1.3 Corollary. *Ergodic S-automorphisms are isomorphic if and only if their underlying permutations are conjugate.*

Proof. Immediate from 1.2 and the application of Ornstein's isomorphism Theorem [6].

Next we shall study the nonergodic automorphisms, that is, those with associated permutations that have finite cycles. To begin with we calculate the entropy of T_ϕ where ϕ is a disjoint product of finite cycles.

1.4 LEMMA. *If ϕ is a product of finite cycles then the Kushnirenko entropy $h_A(T_\phi) = 0$ for all sequences A. In particular $h(T_\phi) = 0$.*

Proof. Let ϕ_j, $j = 1, 2,...$ be the disjoint cycles of ϕ and select $n_j \in \phi_j$. Let $1 \leqslant l_j < \infty$ denote the length of the cycle ϕ_j.

For each j, let $P_0^{(j)}$ denote the partition

$$(\{w: w_{n_j} = i\}: i \in F).$$

If $P^{(j)} = V_{l=0}^{l_j-1} T_{\phi_j}^l P_0^{(j)}$ then $P^{(j)}$ is invariant under T_ϕ for each j. Moreover if $P_l = V_{j=1}^{l} P^{(j)}$ then $P_l \uparrow \epsilon$, and $T_\phi P_l = P_l$ for each l.

Let $A = \{m_k\}$ be any sequence of integers. By Lemma 2 of Kushnirenko [4],

$$h_A(T_\phi) = \lim_{l\to\infty} h_A(T_\phi, P_l).$$

However,

$$\begin{aligned} h_A(T_\phi, P_l) &= \lim_{k\to\infty} \frac{1}{k} H(T_\phi^{m_1}P_l \vee T_\phi^{m_2}P_l \vee \cdots \vee T_\phi^{m_k}P_l) \\ &= \lim_{k\to\infty} \frac{1}{k} H(P_l) = 0 \end{aligned}$$

and it follows that $h_A(T_\phi) = 0$.

1.5 COROLLARY. *T_ϕ has discrete spectrum if and only if ϕ consists entirely of finite cycles.*

Proof. If ϕ consists of finite cycles then 1.4 and Theorem 4 of [4] imply that T_ϕ has discrete spectrum. In case ϕ contains an infinite cycle then T_ϕ has a Bernoulli factor and does not have discrete spectrum.

It is not difficult to see, and this fact was used in Lemma 1.4, that if ϕ consists of disjoint cycles ϕ_1 and ϕ_2 then $T_\phi = T_{\phi_1} \times T_{\phi_2}$. Actually if ϕ is a product of disjoint permutations (i.e., the permutations act on disjoint subsets of the integers) the same result holds. Thus if ϕ is a given permutation and ϕ_1 is the product of all infinite cycles and ϕ_2 the product of the remaining (finite) cycles then $T_\phi = T_{\phi_1} \times T_{\phi_2}$ so that

$$h(T_\phi) = -k \sum_{i\in F} p_i \log p_i,$$

where k is the number of cycles in ϕ_1.

1.6 COROLLARY. *T_ϕ has discrete spectrum iff $h(T_\phi) = 0$, and every S-automorphism is the product of a Bernoulli shift and an automorphism with discrete spectrum.*

2. NONERGODIC AUTOMORPHISMS

In this section we consider those S-automorphisms whose associated permutations do not have any infinite cycles. Three types will be considered determined by the number and lengths of the finite nondegenerate cycles which constitute the associated permutation.

Type 1. The permutations have infinitely many finite cycles with unbounded lengths.

Type 2. The permutations have infinitely many nondegenerate cycles with bounded lengths.

Type 3. The permutations have finitely many finite cycles.

We will show that automorphisms of type 1 are antiperiodic and automorphisms of type 2 are periodic with almost no fixed points. Two isomorphic automorphisms of type 2 are given whose associated permutations are not conjugate, and for certain automorphisms T_ϕ of type 3, any S-automorphism isomorphic to T_ϕ has an associated permutation which is conjugate to ϕ.

2.1 LEMMA. *If ϕ is a finite cycle then*

$$\{w: T_\phi w = w\} = \bigcup_{i\in F} \bigcap_{n\in\phi} \{w: w_n = i\}.$$

2.2 LEMMA. *If ϕ is a product of disjoint cycles ϕ_j, $j = 1, 2, \ldots$, then*

$$\{w: T_\phi w = w\} = \bigcap_j \{w: T_{\phi_j} w = w\}$$

and if follows from Lemma 2.1 *that*

$$\{w: T_\phi w = w\} = \bigcap_j \bigcup_{i\in F} \bigcap_{n\in\phi_j} \{w: w_n = i\}.$$

2.3 LEMMA. *Let ϕ be a cycle of length l and k any positive integer. Let $r = (l, k)$, the greatest common divisor of l and k. Then*

$$\begin{aligned} \{w: T_\phi{}^k w = w\} &= \Omega & &\text{if} \quad k \equiv 0 \ (\mathrm{mod}\ l) \\ &= \bigcap_{j=1}^{r} \left(\bigcup_{i\in F} \bigcap_{n\in\phi_j} \{w: w_n = i\} \right) & &\text{otherwise} \end{aligned}$$

where ϕ_j are disjoint cycles of length l/r.

Proof. This follows from Lemma 2.2 and the easily verified fact that $T_\phi{}^k = T_{\phi^k}$ and ϕ^k consists of r disjoint cycles of length l/r.

2.4 LEMMA. *If ϕ is a product of disjoint cycles ϕ_j then*

$$\{w\colon T_\phi{}^k w = w\} = \bigcap_j \{w\colon T_{\phi_j}^k w = w\}.$$

Proof. This follows from 2.2 since $T_\phi{}^k = T_{\phi^k} = \mathsf{X}_j\, T_{\phi_j}k.$

2.5 PROPOSITION. *If ϕ is a product of disjoint cycles ϕ_j, $j = 1, 2,\ldots, N$ where ϕ_j has length l_j and if r_j = greatest common divisor of l_j and k, then*

$$\mu\{w\colon T_\phi{}^k w = w\} = \prod_{j=1}^N \left(\sum_{i\in F} p_i^{l_j r_j^{-1}}\right)^{r_j}.$$

Proof. Follows from 2.4 and 2.3, the disjointness of the sets in the unions, and the fact that μ is a product measure.

2.6 COROLLARY. *If ϕ contains infinitely many cycles of the same finite length $l > 1$ then $\mu\{w\colon T_\phi w = w\} = 0$.*

Proof. Suppose $\phi = \psi\phi_1\phi_2\phi_3 \cdots$ where ϕ_j has length l for all $j = 1, 2,\ldots.$ Then if $\phi(m) = \phi_1\phi_2 \cdots \phi_m$,

$$\begin{aligned}\mu\{w\colon T_\phi w = w\} &\leqslant \mu\{w\colon T_{\phi(m)} w = w\}\\ &= \left(\sum_{i\in F} p_i{}^l\right)^m \qquad \text{for all } m\end{aligned}$$

and since $l > 1$, $\sum_{i\in F} p_i{}^l < 1$ and $(\sum_i p_i{}^l)^m \to 0$ as $m \to \infty$.

2.7 PROPOSITION. *If ϕ is a product of infinitely many disjoint finite cycles whose lengths are unbounded then T_ϕ is antiperiodic.*

Proof. Let $\{\phi_m\}_{m=1}^\infty$ be the disjoint cycles in ϕ and let the length of ϕ_m be denoted by l_m. By hypothesis $l_m \to \infty$. If w is a fixed point under $T_\phi{}^r$ then it is fixed under $T_{\phi_m}^r$ for every m. Thus if E_r denotes the points of period r under T_ϕ,

$$\mu(E_r) \leqslant \mu\{w\colon T_\phi{}^r w = w\} \leqslant \mu\{w\colon T_{\phi_m}^r w = w\}$$

for all m. From Lemma 2.5 and the fact that $\sum_{i=1}^n x_i{}^r$ subject to the restriction that $\sum_{i=1}^n x_i = 1$, $x_i \geqslant 0$ is maximal at $x_i = 1/n$ for all i it follows that

$$\mu\{w\colon T_{\phi_m}^r w = w\} \leqslant \frac{n^{r_m}}{n^{l_m}} \leqslant \frac{n^r}{n^{l_m}},$$

where $r_m = \text{g.c.d.}\{l_m, r\}$ and n is the cardinality of F. Since $l_m \to \infty$, $\mu\{w: T^r_{\phi_m} w = w\} \to 0$ as $m \to \infty$ and hence $\mu(E_r) = 0$ for every r.

Notice that using Proposition 2.7 it is very easy to construct nonergodic antiperiodic automorphisms with discrete spectrum.

It is easy to see that type 2 automorphisms are periodic. Isomorphism for this type is not determined by the conjugacy class of the underlying permutation. For example, let

$$\phi = \cdots (-2, -1)(0, 1)(2, 3)(4, 5) \cdots$$

and

$$\psi = \cdots (-5, -4)(-3)(-2, -1)(0)(1, 2)(3)(4, 5)(6) \cdots.$$

Then almost every point of Ω is periodic of period two with respect to both T_ϕ and T_ψ and it follows that T_ϕ and T_ψ are isomorphic. (See Theorem 2.8.)

A complete isomorphism invariant can be given for type 3 automorphisms since every point is a periodic point.

2.8 Theorem. *If T and S are metric automorphisms on a nonatomic Lebesque space X and almost every point of X is both S and T periodic then T and S are isomorphic if and only if the measure of the set of T periodic points of period r is equal to the measure of the set of S periodic points of period r for all r.*

Proof. It is clear that if T and S are isomorphic then the measures of the periodic points must be equal.

Suppose that the measures of the sets of periodic points with the same period are equal. Let $E_r(T)$ and $E_r(S)$ denote respectively the T and S periodic points of period r. By hypothesis $X = \bigcup_r E_r(T) = \bigcup_r E_r(S)$ and the unions are disjoint. It is enough to show that T and S are isomorphic when restricted to $E_r(T)$ and $E_r(S)$, respectively.

Let T_r and S_r denote T and S restricted to $E_r(T)$ and $E_r(S)$. Those E_r which have positive measure are nonatomic Lebesque spaces, and the other E_r may be neglected. Thus T_r is a metric automorphism on a Lebesque space all of whose points are periodic of period r and so is S_r. From a theorem of Halmos [2], there exists a set $F_r(T) \subset E_r(T)$ such that $\mu(F_r(T)) = (1/r)\,\mu(E_r(T))$ and the sets $\{T_r^j F_r(T): j = 0, 1, ..., r-1\}$ are disjoint. A similar set $F_r(S) \subset E_r(S)$ also exists. Since $\mu(E_r(T)) = \mu(E_r(S))$ it follows that $\mu(F_r(S)) = \mu(F_r(T))$ and the two Lebesque spaces $F(S_r)$ and $F(T_r)$ are isomorphic. Thus if $\bar{\sigma}_r$ is a metric isomorphism from $F_r(T)$ onto $F_r(S)$, define σ_r on $E_r(T)$ onto $E_r(S)$ by

$$\sigma_r(x) = S_r^{\,i} \bar{\sigma}_r T_r^{-i}(x)$$

for $x \in T_r^{\,i} F(T_r)$, $0 \leqslant i \leqslant r-1$. The disjoint sum of the automorphisms σ_r, $r \geqslant 1$ gives an automorphism σ of X onto X such that $\sigma T = S\sigma$.

2.9 COROLLARY. *If T and S are metric automorphisms on a nonatomic Lebesgue space X almost all of whose points are both T and S periodic then T and S are isomorphic if and only if*

$$\mu\{x: T^r x = x\} = \mu\{x: S^r x = x\}$$

for all $r > 0$.

Proof. It is clear that if T and S are isomorphic then the measures of the fixed point sets under like powers must be equal.

Let $T(r)$ and $S(r)$ denote respectively the measures of the T periodic and S periodic points of period r. It is easy to see that

$$\mu\{x: T^n x = x\} = \sum_{r|n} T(r),$$

so by the Möbus transformation formula (cf. [3, Theorem 266, p. 236])

$$T(n) = \sum_{r|n} m\left(\frac{n}{r}\right) \mu\{x: T^r x = x\}.$$

By hypothesis $\mu\{x: T^r x = x\} = \mu\{x: S^r x = x\}$ for every r so that

$$S(n) = \sum_{r|n} m\left(\frac{n}{r}\right) \mu\{x: S^r x = x\}$$

$$= \sum_{r|n} m\left(\frac{n}{r}\right) \mu\{x: T^r x = x\} = T(n).$$

Thus Theorem 2.8 gives that T and S are isomorphic.

2.10 THEOREM. *If ϕ is a product of k cycles of prime length l and ψ is any permutation such that T_ψ is isomorphic to T_ϕ then ψ is conjugate to ϕ.*

Proof. Almost all points of Ω are periodic under both T_ϕ and T_ψ. It is also true that T_ψ must be periodic of period l. Let ψ consist of cycles of lengths $l_1, l_2, \ldots$. Since T_ψ has period l, and $T_\psi^l = T_{\psi^l}$, ψ has period l and thus the least common multiple of l_j is l. Since l is a prime each l_j must equal l. Thus ψ consists only of cycles of length l. By Lemma 2.5

$$\mu\{w: T_\phi w = w\} = \left(\sum_{i\in F} p_i{}^l\right)^k$$

and by Theorem 2.8

$$\mu\{w: T_\psi w = w\} = \left(\sum_{i\in F} p_i{}^l\right)^k > 0.$$

By Corollary 2.6, ψ consists only of finitely many cycles, say k^1, each of length l. By Theorem 2.7, $(\sum_{i\in F} p_i{}^l)^k = (\sum_{i\in F} p_i{}^l)^{k^1}$ which is true if and only if $k^1 = k$. Thus ψ consists of k cycles of length l and is conjugate to ϕ.

2.11 THEOREM. *If ϕ is a product of k disjoint cycles each of length l^α, where l is a prime, and if ψ is any permutation such that T_ψ is isomorphic to T_ϕ then ψ is conjugate to ϕ.*

Proof. Since T_ψ is isomorphic to T_ϕ, ψ and ϕ have the same order and so the least common multiple of the lengths of the disjoint cycles in ψ is l^α. Thus each cycle has length l^β where $\beta \leqslant \alpha$, and at least one cycle has length l^α. Suppose there are $r \geqslant 1$ cycles in ψ each of length l^α and let the other cycles have lengths l^{α_i} where $\alpha_i < \alpha$ for $i = 1, 2, \ldots, k$. Let $\bar{\alpha} = \max\{\alpha_i: 1 \leqslant i \leqslant k\}$ and $q = l^{\bar{\alpha}}$ so that ψ^q will consist of rq cycles of length $l^\alpha/q = l^{\alpha-\bar{\alpha}}$. Since T_{ϕ^q} is isomorphic to T_{ψ^q} and ϕ^q consists of kq cycles of length l^α/q it follows from Corollary 2.9 that

$$\mu\{w: T_\phi{}^q w = w\} = \mu\{w: T_\psi{}^q w = w\}$$

so that

$$\left(\sum_{i\in F} p_i^{l^\alpha q^{-1}}\right)^{kq} = \left(\sum_{i\in F} p_i^{l^\alpha q^{-1}}\right)^{rq}$$

Thus $kq = rq$ and ψ contains k cycles of length l^α.

Again since T_ψ and T_ϕ are isomorphic Theorem 2.8 implies that

$$\mu\{w: T_\phi w = w\} = \mu\{w: T_\psi w = w\}$$

and using Lemma 2.5 that

$$\left(\sum_{i\in F} p_i^{l^\alpha}\right)^k = \left(\sum_{i\in F} p_i^{l^\alpha}\right)^k \left[\prod_{j=1}^t \left(\sum_{i\in F} p_i^{l^{\alpha_j}}\right)\right]$$

so that

$$\prod_{j=1}^t \left(\sum_{i\in F} p_i^{l^{\alpha_j}}\right) = 1.$$

Since $\sum_{i\in F} p_i^{l^{\alpha_j}} < 1$ unless $l^{\alpha_j} = 1$, it follows that ψ contains only cycles of length l^α and is thus conjugate to ϕ.

2.12 LEMMA. *If ϕ is a single cycle of length pq where p and q are primes and $T_{\phi^q} \cong T_{\psi^q}$ then ψ is either conjugate to ϕ or consists of p cycles of length q and q cycles of length p.*

Proof. Since $T_\psi \cong T_\phi$, $T_\phi r \cong T_\psi r$ for all $r > 0$. In particular $T_{\phi^p} \cong T_{\psi^p}$ and $T_\phi q \cong T_\psi q$. Since ϕ^p and ϕ^q consist of products of disjoint cycles all of the same prime length Theorem 2.10 implies that ψ^q consists of q disjoint cycles of length p and ψ^p consists of p disjoint cycles of length q.

Suppose ψ consists of k cycles of lengths $l_1, l_2, \ldots, l_k$. Let ψ_i be that cycle of length l_i. Assume $q < p$. Since $\psi^p = \psi_1^p \psi_2^p \cdots \psi_k^p$ contains only cycles of length q, each ψ_i^p contains only cycles of length q. Since ψ_i has length l_i, ψ_i^p consists of $gcd(p, l_i)$ cycles of length $l_i/gcd(p, l_i)$. In case $l_i < p$, $gcd(p, l_i) = 1$ and thus $l_i = q$. In case $l_i \geqslant p$, either $gcd(p, l_i) = 1$ or p. In the first case $l_i = q < p$ a contradiction and thus $gcd(p, l_i) = p$. Thus $l_i = n_i p$. However

$$\psi^q = \psi_1^q \cdots \psi_k^q$$

contains only cycles of length p. Thus if $l_i = n_i p$, ψ_i^q contains $gcd(q, n_i p)$ cycles. Since $q < p$, $q < n_i p$ and $gcd(q, n_i p)$ is either q or 1. In case $gcd(q, n_i p) = 1$, ψ_i^q contains one cycle of length $n_i p$ thus $n_i = 1$. In case $gcd = q$, $q \mid n_i p$ and $q \mid n_i$ so that $n_i = m_i q$. Since ψ^q can contain at most q cycles of length p, $m_i = 1$. Thus the only cycles that ψ can contain have lengths pq, p, and q.

In case ψ has a single cycle of length pq any additional cycle of length p will cause ψ^q to have more than q disjoint cycles of length p. A similar contradiction will not allow ψ to have any additional cycle of length q. Thus if ψ contains a cycle of length pq it is a single cycle. In the alternative case ψ consists of p disjoint cycles of length q and q disjoint cycles of length p.

2.13 COROLLARY. *If ϕ consists of a disjoint cycle of length pq when p and q are primes and ψ is any permutation such that $T_\psi \simeq T_\phi$ then ψ is conjugate to ϕ.*

Proof. In case ψ is not conjugate to ϕ, ψ must contain p cycles of length q and q cycles of length p. Thus since $T_\phi \simeq T_\psi$, $\mu\{T_\phi w = w\} = \mu\{T_\psi w = w\}$ implies that

$$\left(\sum_{i\in F} p_i^p\right)^q \left(\sum_{i\in F} p_i^q\right)^p = \sum_{i\in F} p_i^{pq}.$$

However this is impossible since H. D. Brunk [1] has proven that the left hand side is always smaller than the right.

3. THE ZETA FUNCTION OF S-AUTOMORPHISMS

In this section a functions is defined which is a complete isomorphism invariant for S-automorphisms of type 3. It is defined analogously with the zeta function of Artin–Mazur as developed by Smale [7], Williams [10] and others. For another modification of the Artin–Mazur zeta function see [5].

3.1 DEFINITION. Let T_ϕ be a S-automorphism where ϕ consists of a finite number of finite cycles. The zeta function of T_ϕ, denoted by $\zeta(\phi)$, is defined by the equation

$$\zeta(\phi)(z) = \exp\left[-\sum_{k=1}^{\infty} \log \mu\{w\colon T_\phi^k w = w\} \frac{z^k}{k}\right].$$

Notice that since T_ϕ has only finitely many finite cycles $\mu\{w: T_\phi w = w\} > 0$ and each coefficient is finite.

3.2 LEMMA. *If ϕ is a disjoint product of permutations $\{\phi_j\}$ then*

$$\zeta(\phi) = \prod_j \zeta(\phi_j).$$

3.3 THEOREM. *If ϕ and ψ are permutations which consists of finitely many finite cycles then T_ϕ and T_ψ are isomorphic if and only if $\zeta(\phi) = \zeta(\psi)$.*

Proof. Follows immediately from Corollary 2.9.

3.4 LEMMA. *If ϕ is a cycle of length l then*

$$\zeta(\phi) = \prod_{j=1}^{l} (\rho_l{}^j - z)^{\alpha_j}$$

where $\rho_l = \exp(2\pi i/l)$ and

$$\alpha_j = \sum_{k=1}^{l-1} K_k \rho_l^{j(k-1)} \left[\prod_{\substack{i=1\\ i\neq j}}^{l} (\rho_l{}^i - \rho_l{}^j)\right]^{-1}$$

and

$$\begin{aligned} K_k &= -\log\left(\sum_{i\in F} p_i{}^l\right) \qquad \text{if} \qquad k \nmid l \\ &= -k \log\left(\sum_{i\in F} p_i^{l/k}\right) \qquad \text{if} \qquad k \mid l. \end{aligned}$$

Proof. Using Proposition 2.5 one obtains

$$-\log \mu\{w: T_\phi{}^k w = w\} = -(l, k) \log\left[\sum_{i\in F} p_i^{l/(l,k)}\right]$$

so that

$$\begin{aligned} \frac{d}{dz}(\log \zeta(\phi)) &= -\sum_{k=1}^{\infty} (l, k) \log\left[\sum_{i\in F} p_i^{l/(l,k)}\right] z^{k-1} \\ &= K_1 \sum_{k=0}^{\infty} z^{kl} + K_2 \sum_{k=0}^{\infty} z^{kl+1} + \cdots + K_{l-1} \sum_{k=0}^{\infty} z^{kl+(l-2)} \\ &= \frac{\sum_{j=1}^{l-1} K_j z^{j-1}}{1 - z^{j-1}}. \end{aligned}$$

Decomposing this into partial fractions gives

$$\frac{d}{dz}(\log \zeta(\phi)) = \sum_{j=1}^{l} \frac{\alpha_j}{\rho_l{}^j - z} \qquad \text{with} \qquad \rho_l = \exp\left(\frac{2\pi i}{l}\right)$$

and α_j as specified in the lemma.

3.5 Corollary. *If ϕ is a cycle of prime length l then $\zeta(\phi)$ has either a zero or singularity at the* lth *roots of unity and is analytic and nonzero elsewhere.*

Proof. Since l is a prime, K_l is a constant and

$$\frac{d}{dz}(\log \zeta) = \frac{K_j \sum_{j=1}^{l-1} z^j}{1 - z^l}.$$

Since $\sum_{j=1}^{l-1} z^j$ has no zeros at the lth roots of unity $\alpha_j \neq 0$ for all j and each lth root of unity is either a singularity or zero of $\zeta(\phi)$.

3.6 Theorem. *Let ϕ and ψ be permutations which consist of finitely many disjoint cycles of prime lengths. Then T_ϕ and T_ψ are isomorphic if and only if ϕ and ψ are conjugate.*

Proof. It is only necessary to show that if T_ϕ and T_ψ are isomorphic then ϕ and ψ are conjugate. Suppose ϕ consists of k_j disjoint cycles of prime lengths l_j, $j = 1, 2,..., r$ with $l_1 < l_2 < \cdots < l_r$. Using Corollary 3.5 and Lemma 3.2, one obtains

$$\zeta(\phi)(z) = \prod_{j=1}^{r} \prod_{i=1}^{l_j} (\rho_{l_j}^i - z)^{k_j \alpha_i}$$

and none of the $k_j\alpha_i$ are zero. Thus $\zeta(\phi)$ has either singularities or zeros at the l_jth roots of unity, and unless ψ contains cycles of lengths $l_1, l_2, ..., l_r$, $\zeta(\psi)$ will have neither singularities nor zeros at the l_jth roots of unity and hence T_ϕ is not isomorphic to T_ψ by Theorem 3.3. Thus ϕ and ψ must contain cycles of the same lengths.

Suppose ψ contains $k_j' > 0$, cycles of length l_j, $j = 1, 2, ..., r$. Then ψ^{q_i}, where $q_i = \sum_{j \neq i} l_j$, consists of k_i' disjoint cycles of length l_i and ϕ^{q_i} consists of k_i disjoint cycles of length l_i. But T_ϕ and T_ψ are isomorphic so that by Theorem 2.10 $k_i = k_i'$. Thus ϕ and ψ are conjugate.

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